Mathematical modeling and numerical simulation of polythermal glaciers

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Abstract. A mathematical model for polythermal glaciers and ice sheets is presented. The enthalpy balance equation is solved in cold and temperate ice together using an enthalpy gradient method. To obtain a relationship between enthalpy, temperature and water content, we apply a brine pocket parametrization scheme known from sea ice modeling. The proposed enthalpy formulation offers two advantages: (1) the discontinuity at the cold-temperate transition surface is avoided; and (2) no treatment of the transition as an internal free boundary is required. Fourier's law and Fick-type diffusion are assumed for sensible heat flux in cold ice and latent heat flux in temperate ice, respectively. The method is tested on Storglaciären, northern Sweden. Numerical simulations are carried out with a commercial finite element code. A sensitivity study reveals a wide range of applicability and defines the limits of the method. Realistic temperature and moisture fields are obtained over a large range of parameters.

1. Introduction

Polythermal glaciers consist of cold ice (ice below the pressure melting point) and temperate ice (ice at the pressure melting point). Temperate ice is characterized by the existence of a small percentage of liquid water in the ice matrix. The content of liquid water in temperate ice is defined as the mass fraction of water in the ice-water mixture \( \omega = m_w/m \), where \( m_w \) and \( m \) are the mass of water and the mass of the mixture, respectively. The liquid water content varies spatially and temporally in a glacier, but is generally less than 3\%, and, additionally, the maximum liquid water content differs from glacier to glacier [cf. Pettersson et al., 2003, and references therein]. Higher values may be found in fully temperate glaciers. Several studies [Vallon et al., 1976; Murray et al., 2000] indicate that an upper limit for the water content exists above which the percolation of water becomes important.

A first mathematical model for polythermal glaciers and ice sheets was presented by Fowler and Larson [1978]. Hutter [1982] introduced mixture theory for the description of temperate parts and Fowler [1984] discussed the role of salts in ice. Hutter et al. [1988] applied a reduced plane flow model and computed water content at the transition between cold and temperate ice for different moisture diffusivities. A comprehensive overview is given in Greve [1997a].

Polythermal glaciers and ice caps are common at high latitudes, e.g. in the Canadian Arctic [Blatter, 1987; Blatter and Kappenberger, 1988], in Svalbard [Jania et al., 1996] or in Scandinavia [Holmlund and Eriksson, 1989] but can also be found at high altitudes in the Alps [Haeffeli, 1963; Haeberli, 1976]. In ice sheets, temperate zones can be present at the bed. Many present-day ice sheet models, however, ignore the effects of melting, and thus locally violate energy conservation. This “cold-ice method” overestimates the thickness and volume of temperate basal layers, as demonstrated by Greve [1997b] for the Greenland ice sheet. Breuer et al. [2006] modified a three-dimensional “cold-ice method” model to assess the influence of temperate ice in King George Island ice cap. Zwinger et al. [2007] used variational inequalities to impose the constraint that the ice temperature is limited by the melting point. The only truly polythermal ice sheet model SICOPOLIS [Greve, 1995, 1997a, b] solves the Fourier equation for the cold parts and a moisture advection-production equation for the temperate parts together with the jump conditions and kinematic conditions at the free cold-temperate transition surface (CTS).

Numerical approaches to handle phase changes may be divided into two classes: front-tracking methods and enthalpy methods (EM) [Nedjar, 2002]. SICOPOLIS implements a front-tracking method treating the CTS as a moving boundary. This either requires deforming grids or transformed coordinate systems, of which the latter is implemented in SICOPOLIS. Both deforming grids and transformed coordinate systems are somewhat cumbersome to implement.

In this paper, we propose a different strategy to model polythermal glaciers by using an EM which solves the enthalpy equation for the cold and temperate domains together. Enthalpy methods have been used for more than three decades [e.g. Shamsundar and Sparrow, 1975; Voller and Cross, 1981; White, 1982; Voller et al., 1987; Elliott, 1987] and are recommended by many authors owing to their ease of implementation [Nedjar, 2002, and references therein]. In the standard enthalpy method the heat flux is expressed in terms of the temperature gradient. On the other hand, if heat flux is expressed in terms of the gradient in enthalpy, this technique is called the “enthalpy gradient method” (EGM) [Pham, 1995]. Using an EM requires a functional relationship between enthalpy, temperature and water content. To this end, we have chosen to apply a regularization of the enthalpy function known from sea ice modeling.

In the next section, field equations, constitutive equations and boundary conditions for polythermal glaciers are introduced together with the applied regularization. Numerical simulations are described in section 3, and results are presented in section 4. We discuss our results and summarize our conclusions in section 5.

2. Mathematical Model

2.1. Field Equations

Glacier ice is generally assumed to be an incompressible, viscous, heat-conducting non-Newtonian fluid which obeys the Stokes equations,

\[ \nabla \cdot \mathbf{v} = 0, \tag{1} \]
\[ \nabla \cdot \mathbf{T} = -p \mathbf{g}, \]  

(2)

and a balance equation for specific inner energy [Greve, 1997a], \( u \),

\[ \rho u = -\nabla \cdot \mathbf{q} + Q. \]  

(3)

\( \nabla \cdot \) is the divergence operator, \( \mathbf{v} \) is the velocity, \( \mathbf{T} \) is the Cauchy stress tensor, \( \rho \) is the density of ice, \( \mathbf{g} \) is the acceleration due to gravity. \( u \) is the total derivative of \( u \), \( \mathbf{q} \) is the energy flux and \( Q \) is an internal heat source. The specific energy has SI units J kg\(^{-1}\), and the terms in equation (3) have SI units J m\(^{-2}\) s\(^{-1}\) [Moran and Shapiro, 2000]. The stress tensor can be split into an isotropic and a deviatoric part,

\[ \mathbf{T} = -p \mathbf{I} + \mathbf{T}' = -p \mathbf{I} + 2\eta \mathbf{D}, \]  

(4)

where \( p \) is the pressure, \( \mathbf{I} \) and \( \mathbf{T}' \) are the identity and the deviatoric stress tensor, respectively, \( \eta \) is the effective viscosity and \( \mathbf{D} = 1/2 \left( \nabla \mathbf{v} + \nabla \mathbf{v}^T \right) \) is the strain rate tensor.

2.2. Constitutive Equations

The set of field equations (1) to (3) must be closed by constitutive equations for the viscosity \( \eta \) and the energy flux \( \mathbf{q} \). Ice flow

The nonlinear rheology of glacier ice is generally expressed by a power law [Glen, 1955; Steinemann, 1958], relating viscosity \( \eta \) and effective strain rate \( \dot{\varepsilon}_{\text{eff}} \),

\[ \eta = \frac{1}{2} A \dot{\varepsilon}_{\text{eff}}^\eta, \]  

(5)

where \( A \) is the rate factor and \( n \) is the flow exponent. The effective strain rate is

\[ \dot{\varepsilon}_{\text{eff}} = \sqrt{\Pi_2} = \sqrt{\frac{1}{2} \text{tr} (\mathbf{D} \cdot \mathbf{D}),} \]  

(6)

where \( \Pi_2 \) is the second invariant of \( \mathbf{D} \).

The rate factor of temperate ice depends on the water content; however, only one experimental study by Duval [1977] attempted to quantify this relationship. We assume a constant rate factor because in this way a thermomechanically coupled model is avoided as the energy balance equation is decoupled from the Stokes equations.

Thermodynamics

The specific enthalpy, \( H \), is commonly defined as [e.g., Moran and Shapiro, 2000]:

\[ H = u + p/\rho. \]  

(7)

If a material is heated under constant pressure, when there are no volume changes, the enthalpy represents the inner energy [Alexiades and Solomon, 1993]. Because the term "enthalpy method" is widely used in the computational fluid dynamics literature, we will refer to enthalpy instead of inner energy in the remainder of this paper and rewrite equation (3) as

\[ \rho H = -\nabla \cdot \mathbf{q} + Q. \]  

(8)

Ice is defined as cold if a change in enthalpy leads to a change in temperature alone, thus

\[ \delta H = c \delta T, \]  

(9)

where \( c \) is the heat capacity under constant pressure. The energy flux in cold ice, \( \mathbf{q} \), can be described by Fourier’s law,

\[ \mathbf{q} = -k \nabla T, \]  

(10)

where \( k \) is the thermal conductivity.

Ice is defined as temperate if a change in enthalpy leads to a change in liquid water content alone, thus

\[ \delta H = L \delta \omega, \]  

(11)

where \( L \) is the latent heat of fusion. Temperate ice is at the local melting point \( T_m \), although the possibly small dependencies of the melting point on pressure, air-saturation level of the ice and stresses [Harrison, 1972; Kamb, 1972] are neglected in this study. As a consequence the energy flux in temperate ice, \( \mathbf{q} \), is expressed as

\[ \mathbf{q} = L \mathbf{j}. \]  

(12)

where \( \mathbf{j} \) is the diffusive moisture flux [Hutter, 1982; Greve, 1997a]. Little is known about the moisture flux in temperate ice, though Fick-type [Hutter, 1982] or Darcy-type [Fowler, 1984] diffusion have been proposed. A general formulation for the diffusive moisture flux in temperate ice may consider water content, its spatial gradient, deformation and gravity [Hutter, 1982].

2.3. Boundary conditions

Ice flow

At the glacier surface tangential stress vanishes and the resulting boundary condition becomes

\[ \mathbf{T} \cdot \mathbf{n} = p_{\text{at}} \mathbf{n}, \]  

(13)

where \( p_{\text{at}} \) is the atmospheric pressure and \( \mathbf{n} \) is the outward unit normal vector.

Either a basal velocity \( \mathbf{v}_b \) or a given sliding law can be prescribed. If no-slip conditions are assumed, then \( \mathbf{v}_b \equiv 0 \). In case of a sliding law, the basal shear traction \( \tau_b \) and the sliding velocity \( \mathbf{v}_b \) are functionally related through \( \mathbf{F} (\tau_b, \mathbf{v}_b) = 0 \).

Thermodynamics

The thermodynamic boundary conditions at the glacier surface are defined by the energy balance at the surface. However energy fluxes at the surface are the result of a complex climatology and therefore, it is common practice to prescribe Dirichlet conditions for cold and temperate ice, \( T = T_s \) and \( \omega = \omega_s \), respectively. At the cold bed, the geothermal heat flux, \( q_{\text{geo}} \), enters the ice, thus a Neumann condition is applied, \( \mathbf{n} \cdot \mathbf{q} = q_{\text{geo}} \). At the temperate bed, two cases have to be distinguished, either the ice is also temperate above the bed or the ice is cold immediately above the bed. In the first case, all available heat (geothermal and frictional heat) is used for basal melt, thus \( \mathbf{n} \cdot \mathbf{q} = 0 \). In the latter case, part of the heat flux may enter the ice. However, to properly treat this case, the thermal boundary of the domain must be lowered deep enough into the lithosphere below the glacier.

2.4. Enthalpy Gradient Method

For the temperature range in consideration, the dependence of thermodynamic quantities on temperature is small [e.g., Paterson, 1994]. It is thus justified to assume heat capacity and thermal conductivity to be constant. To solve the enthalpy balance equation (8), we express the energy flux \( \mathbf{q} \) in terms of enthalpy. In cold ice, the gradient of equation (9) gives

\[ \nabla T = \frac{1}{c} \nabla H, \]  

(14)

and introducing equation (14) into equation (10) yields the sensible heat flux in cold ice as a function of enthalpy,

\[ \mathbf{q} = -\frac{k}{c} \nabla H. \]  

(15)
As indicated by the name, the “enthalpy gradient method” assumes a enthalpy gradient-driven diffusion and is thus not compatible with Darcy-type diffusion. Therefore, a Fick-type moisture diffusion in temperate ice is assumed and the diffusive moisture flux is then given by

\[ j = -\nu \nabla \omega, \]  

where \( \nu \) is a moisture diffusivity [Hutter, 1982]. In temperate ice, the gradient of equation (11) gives

\[ \nabla \omega = \frac{1}{L} \nabla H. \]  

Introducing equations (16) and (17) into equation (12) yields

\[ q = -\nu \nabla H. \]  

By defining the diffusivity \( \kappa \) as

\[ \kappa = \begin{cases} \frac{\nu}{\rho} & \text{temperate ice} \\ \kappa/(\rho c) & \text{cold ice} \end{cases}, \]  

thus

\[ q = -\rho c \nabla H, \]  

equation (8) can be rewritten as

\[ \rho \left( \frac{\partial H}{\partial t} + \nu \cdot \nabla H \right) = \nabla \cdot (\rho c \nabla H) + Q, \]  

for both cold and temperate ice. The first term is the local rate of change of enthalpy, the second term is enthalpy advection and the third term is enthalpy diffusion. \( Q = \text{tr}(\mathbf{D} \cdot \mathbf{T}) \) is enthalpy production due to strain heating. In this study, we focus on the thermodynamical steady state, thus the enthalpy balance equation reduces to

\[ \rho c \nabla H = \nabla \cdot (\rho \kappa \nabla H) + Q. \]  

### 2.5. Regularization

The enthalpy function of pure water is discontinuous at the pressure melting point. To convert between enthalpy, temperature and water content, a function relating these three quantities is required. We apply a regularization of the enthalpy as a function of temperature to obtain such a relationship. The regularization is applied firstly to the boundary conditions and, secondly, is used to derive temperature and water content from the calculated enthalpy field. Temperature and water content of pure water as a function of enthalpy are shown in Figures 1a and 1b, respectively, together with the applied regularization.

In this work we use a one-sided regularization inspired by the brine pocket parameterization scheme (BPP) known from sea ice modeling [Untersteiner, 1961; Ono, 1967; Magkut and Untersteiner, 1971; Bitz and Lipscomb, 1999; Huwald et al., 2005]. The brine pocket parameterization accounts for the temperature and salinity dependence of the ice properties. The matrix of ice is assumed to consist of freshwater ice and a complex system of cavities between the ice crystals, mostly along the contact lines and points of three and four ice crystals, respectively. These cavities are considered to be filled with a brine solution in thermal equilibrium with the ice. The thermodynamic properties of a mixture of ice and brine define a temperature range in which a change in enthalpy leads to a change both in temperature and in brine content. We define the bulk salinity \( S_i \) of the ice as the specific salinity,

\[ S_i = \frac{m_b}{m_b + m_i}, \]  

where \( m_b, m_i \) and \( m_s \) are the mass of salt, ice and brine, respectively. The specific salinity of the brine solution is defined by

\[ S_b = \frac{m_b}{m_b}, \]  

Furthermore, the specific mass fraction of brine solution in the ice is defined by

\[ \omega_b = \frac{m_b}{m_i + m_b}. \]  

At equilibrium, the ice temperature, \( T_i \), must be equal to the temperature of the liquid brine pockets, \( T_b \); and thus \( T = T_b = T_i \). If this were not the case, the brine pocket would grow or reduce its size thereby adjusting its salinity in such a way that \( T_b = T_i \). The freezing temperature \( T_0 \) is a function of the salinity \( S_b \), which in linear approximation is

\[ T_0(S_b) - T_0 = -\mu S_b, \]  

where \( \mu \) is an empirical constant and \( T_0 = 273.15 \text{K} \) at atmospheric pressure. With equation (25) and the definitions of \( S_i \) and \( S_b \), in equations (23) and (24), we arrive at

\[ \omega_b = -\frac{\mu S_i}{T - T_0} = -\frac{\alpha}{T - T_0}. \]
We take \( S_i \) as a constant and, therefore, \( \mu S_i \) is henceforth replaced by the regularization parameter \( \alpha \) for convenience. The specific enthalpy \( H \) of the ice-brine mixture of mass \( m = m_i + m_b \) can be written as

\[
H = \frac{1}{m} \left[ -L m_i + c_i (T - T_0) m_i + c_b (T - T_0) m_b \right]
\]

where \( c_i \) and \( c_b \) are the heat capacity of ice and brine at \( T_0 \), respectively. The heat capacity of brine is assumed to be that of pure liquid water at the same temperature since the assumed bulk salinity is extremely small. Introducing \( \alpha \), the position of the CTS we use the facts that at the CTS the water content must be zero, \( \omega = 0 \), and the water content is then obtained from equation (27),

\[
H = -L \left[ 1 + \frac{\alpha}{T - T_0} \right] + c_i (T - T_0) + (c_i - c_b) \alpha.
\]

Solving equation (29) for the temperature \( T \), we obtain two solutions

\[
T = \frac{1}{2c_i} \left[ H + L - (c_i - c_b) \alpha \right] \pm \sqrt{\left( H + L - (c_i - c_b) \alpha \right)^2 - 4c_i L \alpha} + T_b.
\]

The minus sign yields the physically meaningful solution. For small salt contents, \( m_w \approx m_i \), thus \( \omega \approx \omega_b \), and the water content is then obtained from equation (27),

\[
\omega = -\frac{\alpha}{T - T_0}.
\]

As a consequence of the application of the BPP, the position of the CTS is not sharply defined anymore. To find the position of the CTS we use the facts that at the CTS temperature must be at the melting point, \( T = T_m \), and the water content must be zero, \( \omega = 0 \) [Hutter, 1982]. To this end we introduce equation (31) into equation (29), which yields

\[
H = -L \left[ 1 - \omega \right] + c_i (T - T_0) (1 - \omega) + c_b (T - T_0) \omega.
\]

In the limits of \( T \to T_m \) and \( \omega \to 0 \) if the CTS is approached from the cold and temperate sides, respectively, we get \( H = -L \) at the CTS from equation (32). This defines the position of the CTS. We therefore redefine

\[
\text{cold ice} \quad \text{if} \quad H < -L,
\]

\[
\text{cold-temperate transition surface} \quad \text{if} \quad H = -L,
\]

\[
\text{temperate ice} \quad \text{if} \quad H > -L.
\]

Finally, the BPP is applied to the thermodynamic boundary conditions. Let \( \Omega \) be the glacier domain and \( \Gamma_c \) its border, where the subscript \( C = B, S \) denotes bed and surface. At the glacier surface, enthalpy is prescribed in terms of temperature (cold ice, \( c \)) or water content (temperate ice, \( t \)). The transformed surface boundary conditions for enthalpy then read:

\[
H = \begin{cases} H_c(T) & \text{on } \Gamma_{S,c} \\ H_t(\omega) & \text{on } \Gamma_{S,t} \end{cases},
\]

with

\[
H_c(T) = -L \left[ 1 + \frac{\alpha}{T - T_0} \right] + c_i (T_S - T_0) + (c_i - c_b) \alpha,
\]

\[
H_t(\omega) = -L (1 - \omega_s) - c_i \frac{\alpha}{\omega_S} + (c_i - c_b) \alpha.
\]

<table>
<thead>
<tr>
<th>Variable or Constant Name</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A ) factor rate ( \cdot )</td>
<td>( 2.22 \cdot 10^{-24} )</td>
<td>Pa(^{-n} ) s(^{-1} )</td>
</tr>
<tr>
<td>( c_b ) specific heat of brine at ( T_m )</td>
<td>( 4.17 \cdot 10^3 )</td>
<td>J kg(^{-1} ) K(^{-1} )</td>
</tr>
<tr>
<td>( c_i ) specific heat of ice at ( T_m )</td>
<td>( 2.008 \cdot 10^3 )</td>
<td>J kg(^{-1} ) K(^{-1} )</td>
</tr>
<tr>
<td>( L ) latent heat of fusion</td>
<td>( 3.34 \cdot 10^7 )</td>
<td>J kg(^{-1} )</td>
</tr>
<tr>
<td>( m ) exponent of the flow law</td>
<td>( -13 )</td>
<td>-</td>
</tr>
<tr>
<td>( \rho ) density of ice</td>
<td>( 900 )</td>
<td>kg m(^{-3} )</td>
</tr>
<tr>
<td>( \rho_{geo} ) geothermal heat flux</td>
<td>( 0.042 )</td>
<td>W m(^{-2} )</td>
</tr>
<tr>
<td>( \alpha ) viscosity regularization</td>
<td>( 10^{-13} )</td>
<td>s(^{-1} )</td>
</tr>
</tbody>
</table>

where \( T_S \) and \( \omega_S \) are the temperature and the water content at the surface, respectively. Equation (34) is obtained by introducing equation (31) into (33). The transformed boundary conditions at the bed read:

\[
\mathbf{n} \cdot \mathbf{q} = q_{geo} \quad \text{if} \quad H < -L,
\]

\[
\mathbf{n} \cdot \mathbf{q} = 0 \quad \text{if} \quad H \geq -L.
\]

3. Numerical Simulations

3.1. Finite Element Method

Numerical solutions based on the Finite Element Method [e.g., Braess, 2007] are obtained using the commercial program package Comsol Multiphysics (www.comsol.com). The glacier domain is approximated by an unstructured triangular mesh. Quadratic Lagrange elements are used for velocity and enthalpy, and linear Lagrange elements are used for pressure; this is Comsol's default setting. Values of constants and parameters used in this study are listed in Table 1.

3.2. Storglaciären

Storglaciären (67°55’N, 18°35’E), northern Sweden, was selected as test glacier. It is a small polythermal glacier located on the eastern side of the Kebnekaise massif in northern Sweden (Figure 2). Storglaciären has a Scandinavian-type (sometimes also called Svalbard-type) thermal structure where most of the ice is temperate, except for a cold surface layer in the ablation zone. The average thickness of the cold layer is 31 m, with a maximum thickness of 65 m along the southern margin. The thickness decreases towards the equilibrium line. The thermal structure of the glacier is known from radar echo soundings [Holmland and Eriksson, 1989; Pettersson et al., 2003]. Using an in-situ calorimetric method, Pettersson et al. [2004] determined absolute values of the water content at the CTS at three thermistor string locations.

A plane flow approximation for Storglaciären along the kinematic center line is used. Figure 2a shows a map of bed and surface topography and the kinematic center line. Figure 2b shows the longitudinal cross section at the kinematic center line, the regions of cold and temperate ice, and the position of the cold-temperate transition surface as measured by Pettersson et al. [2003].
3.3. Ice flow

Aschwanden and Blatter [2005] calculated horizontal and vertical velocities for Storglaciären along the kinematic center line using a longitudinal stress approximation (first order approximation, FOA) developed by Blatter [1995]. The current study solves the full Stokes (FS) equations, but it uses the same parameters (Table 1) and basal velocities (Figure 3) as for the H2 case in Aschwanden and Blatter [2005]. In the two-dimensional case, a shape factor $f$ [Nye, 1965] accounts for the valley shape. The FOA and FS velocity fields are sufficiently similar (Figure 3) to compare the corresponding computed temperature and moisture distributions using the trajectory integration method (TIM) explained in Aschwanden and Blatter [2005], see Figure 4. Trajectories starting at the temperate surface and ending at the CTS are calculated and moisture due to strain heating is integrated along these trajectories.

A basal velocity $v_b$ is prescribed at the glacier base. The dotted line in Figure 3 is the horizontal component of the imposed basal velocity. Values are in meters per year.

Aschwanden and Blatter (2005) (dashed line, first order approximation) and this study (solid line, full Stokes). Dotted line is the horizontal component of the imposed basal velocity. Values are in meters per year.

3.4. Thermodynamics

The diffusivity is a step function with a step at the CTS from $k_i/(\rho c_i)$ to $\nu/\rho$, where $k_i$ and $c_i$ are thermal conductivity and heat capacity of ice at $T_{\text{ref}}$, respectively. This step is implemented with a smoothed Heaviside function with a continuous first derivative (a piecewise polynomial of degree three), spreading the step over a range of $1500 \text{ J kg}^{-1}$.

The transition between cold and temperate surface conditions is gradual and hence, a smoothed Heaviside function with a smoothing width of 70 m was applied. The transition occurs at a distance of approximately 1670 m from the bergschrund [Pettersson et al., 2003].

A smoothed Heaviside function is applied to spread the thermal transition from temperate to cold ice at the bed over an enthalpy range of 100 J kg$^{-1}$.

The steady-state enthalpy problem is then defined by the following equations:

$$\rho v \cdot \nabla H = \nabla \cdot (\rho c v H) + Q \quad \text{in } \Omega \nabla \cdot H = H_c(T) \quad \text{on } \Gamma_{S,c}$$

$$\n \cdot q = 0 \quad \text{on } \Gamma_{B,1} \quad \n \cdot q = q_{\text{geo}} \quad \text{on } \Gamma_{B,c}$$

The nonlinear system of equations (37) is solved by Comsol Multiphysics using an affine invariant form of the damped Newton method [Deufhard, 1974], which is the programs’ default solver for stationary nonlinear problems. From the simulated enthalpy distribution, temperature and moisture content are then obtained from equations (30) and (31).

3.5. Role of advection and diffusion

The dimensionless Peclet number is a measure of the relative importance of advection to diffusion; the higher the Peclet number, the more important is advection. Values larger than one indicate that the problem is advection-dominated. The global Peclet number, $Pe = \bar{U} L / D$, can be estimated from typical values for velocity $\bar{U}$, length scale $L$, and diffusivity $D$. The global Peclet number can thus be used to estimate the importance of advection.
\[ P_{Ch} = \frac{v h}{\kappa}, \]  
where \( h \) is the length of the longest edge of an element. The local Peclet number indicates areas where either advection or diffusion is the dominant process.

### 3.6. Numerical experiments

A sensitivity study is carried out to assess the relative importance of selected model parameters. The regularization parameter \( \alpha \), the moisture diffusivity \( \nu \), water content at the temperate surface \( \omega_S \), temperature at the cold surface \( T_S \) and maximum element size \( h \) are tested to probe the limits of the enthalpy gradient method. A set of parameters is chosen as a control run \( \text{ctrl} \), the values of which are typed in bold face in Table 2, and are used in all simulations unless stated otherwise. Due to the BPP it is not possible to prescribe a temperate surface water content \( \omega_S = 0 \, \text{g kg}^{-1} \), instead a small number \( 10^{-2} \, \text{g kg}^{-1} \) is used.

### 4. Results

Simulated enthalpy, derived temperature, derived water content, and local Peclet number for the \( \text{ctrl} \) run are shown in Figure 5. For convenience, temperature is given in degree Celsius, \( \theta = T - T_0 \). Figure 5b reveals that advection is the dominant transport process in most of the temperate ice, while diffusion prevails in cold ice and in temperate ice near the bergschrund. The mean thickness of the cold surface layer is 38 m.
Figure 5. Enthalpy (a), local Peclet number (b), temperature (c), and water content (d) for the control run \textit{ctrl}. The dashed line marks the position of the calculated CTS. Values are in joule per kilogram (enthalpy), unitless (Peclet number), degree Celsius (temperature), and grams water per kilogram mixture (water content).

To investigate the influence of mesh size on the solution, both the Stokes equations and the enthalpy equation are solved on meshes with an maximum element size of 5 m, 10 m, 20 m and 40 m. Results (not shown) indicate that the size of the triangular elements does not significantly affect the calculated position of the CTS.

Little is known about the possible range of the moisture diffusion coefficient. Applied values are in the range $10^{-2} - 10^{-6}$ kg m$^{-1}$ s$^{-1}$ [Hutter et al., 1988; Greve [1995, 1997a] assumes a vanishing diffusive moisture flux, although, for numerical stability, uses $\nu = 10^{-6}$ kg m$^{-1}$ s$^{-1}$. In this study, the range of moisture diffusivities for which stable numerical solutions could be obtained is $10^{-2} - 10^{-4}$ kg m$^{-1}$ s$^{-1}$. Figure 6 shows

Figure 6. Position of the CTS for different moisture diffusivities $\nu$.

Figure 7. Range of applicability of the brine pocket parametrization scheme. Absolute values of minimum temperature and maximum water content are good indicators.
CTS positions for selected moisture diffusivities. For moisture diffusivities smaller than $5 \cdot 10^{-3} \text{kg m}^{-1} \text{s}^{-1}$, the position of the CTS does not vary significantly.

The viability of the brine pocket parameterization strongly depends on an appropriate choice of the regularization parameter $\alpha$. Here, we explore the parameter range for which the BPP produces reliable results. Runs were performed from $\alpha = 10^{-20}$ to $10^{-1} \, ^\circ\text{C}$ in steps of factor ten. Minimum and maximum temperatures are given at the cold surface temperature and by the melting point, respectively. Due to the BPP, the melting point can only be reached within a finite limit. Minimum moisture content occurs at the temperate surface but maximum water content depends on strain heating. Figure 7 displays absolute values of minimum temperature and maximum water content as a function of the regularization parameter. Absolute values of minimum temperature are constant from $\alpha = 10^{-20}$ to $10^{-5} \, ^\circ\text{C}$ and then start to increase. The maximum water content is stable between $\alpha = 10^{-17}$ and $10^{-5} \, ^\circ\text{C}$.

The mean thickness of the cold surface layer $H_c$ decreases with both increasing water content at the temperate surface (Figure 8a) and increasing temperature at the cold surface (Figure 8b). $H_c$ decreases from 32 m for $\omega_S = 2 \text{ g}_w \text{ kg}^{-1}$ to 16 m for $\omega_S = 20 \text{ g}_w \text{ kg}^{-1}$ and from 42 m for a surface temperature $\theta_S = -7 \, ^\circ\text{C}$ to 20 m for $\theta_S = -1 \, ^\circ\text{C}$. The decrease of the thickness of the cold surface layer with increasing water content and temperature at the surface is consistent with the findings of Pettersson et al. [2007] who investigated the sensitivity of Storglaciären’s cold surface layer to different forcing parameters.

5. Discussion and Conclusions

The temperature and moisture fields (Figure 5c,d) derived from the calculated enthalpy distribution (Figure 5a) clearly demonstrate that the enthalpy gradient method (EGM) is capable of simulating a Scandinavian-type thermal structure. It is not the aim of this study to compare simulated and measured CTS positions but to demonstrate the applicability of the EGM to the Scandinavian-type thermal structure.

The system of equations (37) define an advection-diffusion-production problem which is parabolic, but becomes hyperbolic if diffusion is neglected. Very small numerical values for the diffusion coefficient result in an advection-dominated transport problem which is very nearly ill-conditioned. We address the advection-domination through a fine grid resolution with target of roughly constant a posteriori values for $Pc_b$ might be favorable. In this case, for decreasing moisture diffusivities, the maximum element size must decrease as well. The solver did not converge for moisture diffusivities smaller than $10^{-4} \text{ kg m}^{-1} \text{s}^{-1}$. This may be attributed to a high local Peclet number but not to non-Newtonian viscosity effects because the enthalpy equation was solved independently from the Stokes equations. However our results are not conclusive.

The moisture field computed with the EGM, using the smallest moisture diffusivity for which results could be obtained, $\nu = 10^{-4} \text{ kg m}^{-1} \text{s}^{-1}$, coincides well with the result of the trajectory integration method (TIM) (Figure 9). Both fields were computed with the same full Stokes velocity field. This suggests that for vanishing moisture diffusion the EGM should be equivalent to the TIM, though we have no mathematical proof for this. Thus, the EGM offers a viable alternative to trajectory integration. However, the trajectory integration method is only applicable if the thermal structure is known a-priori, e.g. from measurements.

The regularization scheme applies the brine pocket parameterization scheme used in sea ice models. The regularization parameter $\alpha$ depends on the bulk salinity $S_i$ and defines the degree of regularization. Values of $\alpha$ less than $10^{-15} \, ^\circ\text{C}$ are too close to machine precision and produce numerical artifacts. Values larger than $10^{-5} \, ^\circ\text{C}$ influence the resulting fields in a physically unrealistic way. This may be related to our choice of $\omega_S = 10^{-5} \text{ g}_w \text{ kg}^{-1}$. As a result of applying the BPP, zero water content and a temperature of zero degrees Celsius are undefined, and hence, a small non-zero value must be prescribed instead. The regularization

![Figure 8](attachment:image.png)

Figure 8. Position of the CTS for selected temperate surface water contents $\omega_S$ (a) and cold surface temperatures $\theta_S$ (b).
parameter should then be an order of magnitude smaller than $\omega_S$. Temperature as a function of the regularization parameter is stable over almost the whole tested range except for the largest values of $\alpha$. The moisture content is more sensitive to the choice of $\alpha$ than the temperature field. Remarkably, for $\omega_S = 10^{-5} \text{kg} \text{water} \text{kg}^{-1}$, the moisture content and temperature fields are invariant over almost ten orders of magnitude of $\alpha$, a fact that supports the application of the regularization method.

We have demonstrated the feasibility of the enthalpy gradient method for 2-dimensional steady state Scandinavian-type polythermal glaciers. This is a first step to a comprehensive thermodynamical model of glaciers and ice sheets, and points the way to future model developments: (1) testing the method for different polythermal structures in glaciers, (2) for transient enthalpy fields, (3) for thermomechanically coupled simulations, and (4) for 3-dimensional situations. The enthalpy gradient method solves a field equation of a similar mathematical form as the Fourier equation for temperature, which is applied in most ice sheet models. This is an important advantage for the inclusion of the enthalpy gradient method in existing ice sheet models, and it allows us to convert “cold-ice method” ice sheet models into polythermal ice sheet models and to couple both moisture content and temperature fields to rheological properties of the ice with relatively minor modifications. Additionally, the corresponding type of polythermal structure should result uniquely from the imposed boundary conditions, thus, no extra handling of different cases and tracking of the cold temperature transition surface should be required with the enthalpy gradient method.

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Figure 9. Content of liquid water simulated by TIM (solid line) and EGM (dotted line). Dashed line indicates position of the CTS as calculated by the EGM. A moisture diffusivity of $\nu = 10^{-4} \text{kg m}^{-2} \text{s}^{-1}$ was used for EGM. Both TIM and EGM use the FS velocity field. Values are in grams water per kilogram mixture.


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